

# WIDTHS OF HIGHLY EXCITED SHAPE RESONANCES

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**ABSTRACT.** We study the widths of shape resonances for the semiclassical multi-dimensional Schrödinger operator, in the case where the frequency remains close to some value strictly larger than the bottom of the well. Under a condition on the behavior of the resonant state inside the well, we obtain an optimal lower bound for the widths.

*Keywords:* Resonances; life-time; semiclassical.

*Subject classifications:* 35P15; 35C20; 35S99; 47A75.

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## 1. INTRODUCTION

The study of shape resonances is a rather old subject in semiclassical analysis, and since the years 80's many mathematical works has been done in order to both locate them and estimate their widths (see, e.g., [AsHa, CDKS, HeSj2, HiSi, FLM] and references therein). In particular, one should mention the work [CDKS], where the existence of shape resonances exponentially close to the real axis is proved, and the work [HeSj2], where a more refined analysis leads to optimal estimates on the widths of resonances that are near a local minimum of the potential. For more excited shape resonances, however, only lower bounds on their widths are available in general, except for the one-dimensional case where the exact asymptotic behavior can be determined : see [Se].

As it is well known, the physical interest of such studies relies on the fact that the widths of the resonances rare directly related to the life-time of metastable quantum states.

The purpose of this work is to extend some of the results of [Se] to the multidimensional case.

More precisely, considering the semiclassical Schrödinger operator  $P := -h^2\Delta + V(x)$  on  $L^2(\mathbb{R}^n)$  with  $n \geq 1$ , we plan to produce optimal exponential estimates on the widths of highly excited shape resonances, that is, shape resonances that tend to an energy  $E_0$  greater than the local minimal of the potential  $V$ . In contrast with [Se], here we assume that the potential well (that is, the bounded component  $U$  of  $\{V \leq E_0\}$ ) is connected, excluding the situation of possible interacting wells. In this situation, the general multidimensional result says that any resonance  $\rho = \rho(h)$  that tends to  $E_0$  as  $h \rightarrow 0_+$  is such that, for any  $\varepsilon > 0$ , one has,

$$|\operatorname{Im} \rho| \leq \mathcal{O}(e^{-(2S_0-\varepsilon)/h})$$

uniformly as  $h \rightarrow 0_+$ . Here,  $S_0 > 0$  is the Agmon distance (that is, the degenerate distance associated with the pseud-metric  $\max(V - E_0, 0)dx^2$ ) between  $U$  and the unbounded component  $\mathcal{M}$  of  $\{V \leq E_0\}$ .

In other words  $\rho$  satisfies,

$$(1.1) \quad \limsup_{h \rightarrow 0_+} h \ln |\operatorname{Im} \rho| \leq -2S_0.$$

When  $n = 1$ , this result is improved into (see [Se], Theorem 0.2),

$$(1.2) \quad \lim_{h \rightarrow 0_+} h \ln |\operatorname{Im} \rho| = -2S_0.$$

Here we plan to extend this improvement to the multidimensional case. Because of (1.1), all we need to prove is that, for any  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon > 0$  such that,

$$(1.3) \quad |\operatorname{Im} \rho| \geq \frac{1}{C_\varepsilon} e^{-(2S_0 + \varepsilon)/h},$$

for all  $h > 0$  small enough.

In order to produce such a good lower bound, when  $n \geq 2$  it is necessary to add an assumption on the size of the resonant state inside  $U$ . This assumption is actually implied by a geometric condition on the classical Hamilton flow above  $U$  (see Remark 4.4) that is automatically satisfied in the one-dimensional case. Roughly speaking, this condition says that the energy shell  $\Sigma_{E_0} := \{(x, \xi) \in \mathbb{R}^{2n}; \xi^2 + V(x) = E_0\}$  is sufficiently well covered by the classical Hamilton flow, in the sense that any open set intersecting  $\Sigma_{E_0}$  is flowed over a whole neighborhood of  $\Sigma_{E_0}$  (this can be understood as a kind of ergodicity of the flow on  $\Sigma_{E_0}$ ).

From a technical point of view, this problem is very close to that of estimating the tunneling for a symmetric double-wells at high excited energies, as considered e.g. in [Ma1] (and indeed, part of our argument will use the results of [Ma1]). However, an additional difficulty comes from the fact that here, the quantity we have to study mainly involves the size of the resonant state in  $\mathcal{M}$ . This means that our work will essentially consist in connecting the size of the resonant state in  $\mathcal{M}$  to that in  $U$ , through the barrier  $\mathcal{B} := \{V > E_0\}$ . The results of [Ma1] permits us to connect the size of the state in  $\mathcal{B}$  to that in  $U$  only, but not its size in  $\mathcal{M}$  to that in  $\mathcal{B}$ . Indeed, it appears that the argument of [Ma1] (which is typically an argument of propagation of microlocal analyticity) does not seem easy to adapt for this last step. However, following an idea already present in [DGM], one can develop some explicit Carleman-type inequalities that permits us to cross the border between  $\mathcal{M}$  and  $\mathcal{B}$ , and to conclude.

## 2. NOTATIONS AND ASSUMPTIONS

We study the spectral properties near energy 0 of the semiclassical Schrödinger operator,

$$P := -h^2\Delta + V(x)$$

on  $L^2(\mathbb{R}^n)$ , where  $x = (x_1, \dots, x_n)$  is the current variable in  $\mathbb{R}^n$  ( $n \geq 1$ ),  $h > 0$  denotes the semiclassical parameter, and  $V$  represents the potential energy.

We assume,

**Assumption 1.** *The potential  $V$  is smooth and bounded on  $\mathbb{R}^n$ , and it satisfies,*

- $\{V \leq 0\} = U \cup \mathcal{M}$  where  $U$  is compact and connected, and  $U \cap \mathcal{M} = \emptyset$ ;
- $V$  has a strictly negative limit  $-L$  as  $|x| \rightarrow \infty$ .

This typically describes the situation where so-called shape resonances appear. In order to be able to define such resonances, we also assume,

**Assumption 2.** *The potential  $V$  extends to a bounded holomorphic functions near a complex sector of the form,  $\mathcal{S}_\delta := \{x \in \mathbb{C}^n ; |\operatorname{Im} x| \leq \delta |\operatorname{Re} x|\}$ , with  $\delta > 0$ . Moreover  $V$  tends to its limit at  $\infty$  in this sector.*

We also assume,

**Assumption 3.**  *$E = 0$  is a non-trapping energy for  $V$  above  $\mathcal{M}$ .*

The fact that 0 is a non-trapping energy for  $V$  above  $\mathcal{M}$  means that, for any  $(x, \xi) \in p^{-1}(0)$  with  $x \in \mathcal{M}$ , one has  $|\exp tH_p(x, \xi)| \rightarrow +\infty$  as  $t \rightarrow \infty$ , where  $p(x, \xi) := \xi^2 + V(x)$  is the symbol of  $P$ , and  $H_p := (\nabla_\xi p, -\nabla_x p)$  is the Hamilton field of  $p$ . It is equivalent to the existence of a function  $F \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$ , supported near  $\{p = 0\} \cap \{x \in \mathcal{M}\}$ , that satisfies,

$$(2.1) \quad H_p F(x, \xi) > 0 \text{ on } \{p = 0\}.$$

It also implies that  $\mathcal{M}$  has a smooth boundary on which  $\nabla V \neq 0$ .

For our purpose, we will also need an additional geometric assumption (that we believe to be generic).

We denote by  $d_V$  the so-called Agmon distance associated with  $V$ , that is, the pseudo-distance associated with the pseudo-metric  $\max(0, V)dx^2$ . We also denote by  $G$  the set of all minimal geodesics (relatively to the Agmon distance  $d_V$ ) between  $U$  and  $\mathcal{M}$  that meet each boundary  $\partial U$  and  $\partial \mathcal{M}$  at one point only. We assume,

**Assumption 4.**  *$G$  is a finite set.*

Note that this assumption is probably purely technical only, and can hopefully be removed by refining our construction and by using a convenient partition of unity.

In the rest of the paper, we set,

$$S_0 := d_V(U, \mathcal{M}).$$

Thanks to our assumptions, we necessarily have  $S_0 > 0$ .

### 3. RESONANCES

In the previous situation, the essential spectrum of  $P$  is  $[-L, +\infty)$ . The resonances of  $P$  can be defined by using a complex distortion in the following way (see, e.g., [Hu]): Let  $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$  such that  $f(x) = x$  for  $|x|$  large enough. For  $\theta \neq 0$  small enough, we define the distorted operator  $P_\theta$  as the value at  $\nu = i\theta$  of the extension to the complex of the operator  $U_\nu P U_\nu^{-1}$  which is defined for  $\nu$  real, and analytic in  $\nu$  for  $\nu$  small enough, where we have set,

$$(3.1) \quad U_\nu \phi(x) := \det(1 + \nu df(x))^{1/2} \phi(x + \nu f(x)).$$

By using the Weyl Perturbation Theorem, one can also see that the essential spectrum of  $P_\theta$  is given by,

$$\sigma_{ess}(P_\theta) = e^{-2i\theta} \mathbb{R}_+.$$

It is also well known that, when  $\theta$  is positive, the discrete spectrum of  $P_\theta$  satisfies,

$$\sigma_{disc}(P_\theta) \subset \{\operatorname{Im} z \leq 0\}.$$

Then, those eigenvalues of  $P_\theta$  that are located in the complex sector  $\{\operatorname{Re} z > 0; -2\theta < \arg z \leq 0\}$  are called the resonances of  $P$  [Hu, HeSj2, HeMa], they form a set denoted by  $\operatorname{Res}(P)$  (on the contrary, when  $\theta < 0$ , the eigenvalues

of  $P_\theta$  are just the complex conjugates of the resonances of  $P$ , and are called anti-resonances).

If  $\rho$  is a resonance, the quantity  $|\operatorname{Im} \rho|$  is called the width of  $\rho$ , and its physical importance comes from the fact that its inverse  $|\operatorname{Im} \rho|^{-1}$  corresponds to the life-time of the corresponding resonant state.

Let us observe that the resonances of  $P$  can also be viewed as the poles of the meromorphic extension, from  $\{\operatorname{Im} z > 0\}$ , of some matrix elements of the resolvent  $R(z) := (P - z)^{-1}$  (see, e.g., [ReSi, HeMa]).

It is proved in [HeSj1, HeSj2] that, in this situation, the resonances of  $P$  near 0 are close to the eigenvalues of the operator

$$(3.2) \quad \tilde{P} := -h^2 \Delta + \tilde{V}$$

where  $\tilde{V} \in C^\infty(\mathbb{R}^n; \mathbb{R})$  coincides with  $V$  in  $\{\operatorname{dis}(x, M) \geq \delta\}$  ( $\delta > 0$  is fixed arbitrarily small), and is such that  $\inf_{\{\operatorname{dis}(x, M) \leq \delta\}} \tilde{V} > 0$ . The precise statement is the following one : Let  $I(h)$  be a closed interval containing 0, and  $a(h) > 0$  such that  $a(h) \rightarrow 0$  as  $h \rightarrow 0_+$ , and, for all  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  satisfying,

$$(3.3) \quad a(h) \geq \frac{1}{C_\varepsilon} e^{-\varepsilon/h};$$

$$(3.4) \quad \sigma(\tilde{P}) \cap ((I(h) + [-2a(h), 2a(h)]) \setminus I(h)) = \emptyset,$$

for all  $h > 0$  small enough. Then, there exists a constant  $\varepsilon_1 > 0$  and a bijection,

$$\tilde{\beta} : \sigma(\tilde{P}) \cap I(h) \rightarrow \operatorname{Res}(P) \cap \Gamma(h),$$

where we have set,

$$\Gamma(h) := (I(h) + [-a(h), a(h)) + i[-\varepsilon_1, 0],$$

such that, for any  $\varepsilon > 0$ , one has,

$$(3.5) \quad \tilde{\beta}(\lambda) - \lambda = \mathcal{O}(e^{-(2S_0 - \varepsilon)/h}),$$

uniformly as  $h \rightarrow 0_+$ .

In particular, since the eigenvalues of  $\tilde{P}$  are real, one obtains that, for any  $\varepsilon > 0$ , the resonances  $\rho$  in  $\Gamma(h)$  satisfy,

$$(3.6) \quad \operatorname{Im} \rho = \mathcal{O}(e^{-(2S_0 - \varepsilon)/h}).$$

From now on, we consider the particular case where  $I(h)$  consists of a unique value  $E(h)$ , such that,

$$\begin{aligned}
 (3.7) \quad & E(h) \in \sigma_{disc}(\tilde{P}); \\
 & E(h) \rightarrow 0 \text{ as } h \rightarrow 0_+; \\
 & \sigma(\tilde{P}) \cap [E(h) - 2a(h), E(h) + 2a(h)] = \{E(h)\}, \\
 & \text{where } a(h) \text{ satisfies (3.3).}
 \end{aligned}$$

We denote by  $u_0$  the normalised eigenstate of  $\tilde{P}$  associated with  $E(h)$ , and, applying (3.5), we also denote by  $\rho = \rho(h)$  the unique resonance of  $P$  such that  $\rho - E(h) = \mathcal{O}(e^{-(2S_0 - \varepsilon)/h})$ .

The purpose of this paper is to obtain a lower bound on the width  $|\operatorname{Im} \rho|$ , possibly of the same order of magnitude as the upper bound.

#### 4. MAIN RESULT

Following the ideas of [Ma1], we consider the following additional assumption of non degeneracy,

**Assumption [ND]** For all  $\varepsilon > 0$  and for all neighborhood  $W$  of the set  $\bigcup_{\gamma \in G} (\gamma \cap \partial U)$ , there exists  $C = C(\varepsilon, W) > 0$  such that, for all  $h > 0$  small enough, one has,

$$\|u_0\|_{L^2(W)} \geq \frac{1}{C} e^{-\varepsilon/h}.$$

Our main result is,

**Theorem 4.1.** *Suppose Assumptions 1-3, (3.7), and Assumption [ND]. Then, for any  $\varepsilon > 0$ , there exists  $C(\varepsilon) > 0$  such that, for all  $h > 0$  small enough, one has,*

$$(4.1) \quad |\operatorname{Im} \rho(h)| \geq \frac{1}{C(\varepsilon)} e^{-(2S_0 + \varepsilon)/h}.$$

**Remark 4.2.** *In view of (3.6), this lower bound is optimal. Indeed, a consequence of (3.6) and (4.1) is the following identity:*

$$(4.2) \quad \lim_{h \rightarrow 0_+} h \ln |\operatorname{Im} \rho| = -2S_0$$

**Remark 4.3.** *As in [Ma1], it can be shown that Assumption [ND] is indeed necessary to have (4.1), and examples where Assumption [ND] is not satisfied can be constructed.*

**Remark 4.4.** *Assumption [ND] is always satisfied in the one dimensional case. When  $n \geq 2$ , thanks to standard properties of propagation of the microsupport (see, e.g., [Ma2]), a sufficient condition to have Assumption [ND] is the following geometrical one (see also [Ma1]): For any neighborhood  $W$  in  $\mathbb{R}^{2n}$  of  $\bigcup_{\gamma \in G} (\gamma \cap \partial U) \times \{0\}$ , the set  $\bigcup_{t \in \mathbb{R}} \exp tH_p(W)$  is a neighborhood of  $\Sigma_0 := \{\xi^2 + V(x) = 0, x \in U\}$ . Obviously, an equivalent simpler formulation is: For any open set  $W$  intersecting  $\Sigma_0$ ,  $\bigcup_{t \in \mathbb{R}} \exp tH_p(W)$  is a neighborhood of  $\Sigma_0$ .*

## 5. REDUCTION TO AN ESTIMATE IN $\mathcal{M}$

From now on, we denote by  $u$  the resonant state of  $P$  associated with the resonance  $\rho$ , and normalised in such a way that,

$$(5.1) \quad \|u\|_{L^2(\ddot{O})} = 1,$$

where  $\ddot{O} := \mathbb{R}^n \setminus \mathcal{M}$ . Then, it is well known (see, e.g., [HeSj2]) that for any bounded set  $\mathcal{B} \subset \mathcal{M}$ , and for any  $\varepsilon > 0$ , one has,

$$(5.2) \quad \|u\|_{L^2(\mathcal{B})} = \mathcal{O}(e^{-(S_0 - \varepsilon)/h}).$$

In addition, if we set,

$$\mathcal{T}_1 := \bigcup_{\gamma \in G} (\gamma \cap \partial \mathcal{M})$$

(the so-called set of “points of type 1” in the terminology of [HeSj2]), and if  $\mathcal{B}$  stays away from the set,

$$\mathcal{A} := \Pi_x \left( \bigcup_{t \in \mathbb{R}} \exp tH_p(\mathcal{T}_1 \times \{0\}) \right),$$

(where  $\Pi_x$  stands for the natural projection  $(x, \xi) \mapsto x$ , and  $H_p := (\partial_\xi p, -\partial_x p)$  is the Hamilton field of  $p(x, \xi) := \xi^2 + V(x)$ ), then there exists  $\varepsilon_0 > 0$  and a neighborhood  $\mathcal{B}'$  of  $\mathcal{B}$  such that,

$$(5.3) \quad \|u\|_{L^2(\mathcal{B}')} = \mathcal{O}(e^{-(S_0 + \varepsilon_0)/h}).$$

On the other hand, performing Stokes formula on an open domain  $\Omega \supset \ddot{O}$ , we see as in [HeSj2], Formula (10.65), that one has,

$$(5.4) \quad (\text{Im } \rho) \|u\|_{L^2(\Omega)}^2 = -h^2 \text{Im} \int_{\partial \Omega} \frac{\partial u}{\partial \nu} \bar{u} ds,$$

where  $ds$  is the surface measure on  $\partial \Omega$ , and  $\nu$  stands for the outward pointing unit normal to  $\Omega$ . Using (5.2)-(5.4), we deduce that, for some  $\varepsilon'_0 > 0$ , one



has,

$$(5.5) \quad \operatorname{Im} \rho = -h^2 \operatorname{Im} \int_{\partial\Omega \cap \mathcal{A}'} \frac{\partial u}{\partial \nu} \bar{u} ds + \mathcal{O}(e^{-(2S_0 + \varepsilon'_0)/h}),$$

where  $\mathcal{A}'$  is an arbitrarily small neighborhood of  $\mathcal{A}$ .

In order to transform this expression into a more practical one, we plan to use the analytic pseudodifferential calculus introduced in [Sj]. For this purpose, we first have to prove some a priori estimate on  $u$  near  $\mathcal{A}$ .

So, let  $z_1 \in \mathcal{T}_1$ , let  $W_1$  be a neighborhood of  $z_1$  in  $\partial\mathcal{M}$ , and for  $t_0 > 0$  sufficiently small, consider the two Lagrangian manifolds,

$$\Lambda_{\pm} := \bigcup_{0 < \pm t < 2t_0} \exp t H_p(W_1 \times \{0\}) \quad (\subset \{p = 0\}).$$

(Note that they are Lagrangian because  $W_1 \times \{0\}$  is isotropic.) Then, it is easy to check that  $\Lambda_{\pm}$  projects bijectively on the base, and since  $p(x, \xi)$  is an even function of  $\xi$ , we see that they can be represented by an equation of the type,

$$\Lambda_{\pm} : \xi = \pm \nabla \psi(x),$$

where  $\psi$  is a real-analytic function, such that,

$$(5.6) \quad (\nabla \psi(x))^2 + V(x) = 0.$$

Now, we set  $z_0 := \Pi_x \exp t_0 H_p(z_1, 0)$ , and we still denote by  $\psi$  an holomorphic extension of  $\psi$  to a complex neighborhood of  $z_0$ . We have,

**Proposition 5.1.** *For any  $\varepsilon_1 > 0$ , one has,*

$$e^{-i\psi/h} u \in H_{-S_0 + \varepsilon_1 |\operatorname{Im} x|, z_0},$$

where  $H_{-S_0 + \varepsilon_1 |\operatorname{Im} x|, z_0}$  is the Sjöstrand's space consisting of  $h$ -dependent holomorphic functions  $v = v(x; h)$  defined near  $z_0$ , such that, for all  $\varepsilon > 0$ ,

$$v(x, h) = \mathcal{O}(e^{(-S_0 + \varepsilon_1 |\operatorname{Im} x| + \varepsilon)/h}),$$

uniformly for  $x$  close to  $z_0$  and  $h > 0$  small enough.

*Proof.* Set

$$(5.7) \quad v(x, h) := e^{-i\psi/h + S_0/h} u(x, h).$$

We have to prove that  $v \in H_{\varepsilon_1 |\operatorname{Im} x|, z_0}$  for all  $\varepsilon_1 > 0$ .

Let  $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  supported in a small neighborhood of  $z_0$ , and such that  $\chi = 1$  near  $z_0$ . Setting  $\varphi(x, y, \tau) := (x - y)\tau + \frac{1}{2}i(x - y)^2$  and  $a(x, \tau) := 1 + \frac{1}{2}ix\tau$ , we can write (see, e.g., [Sj]),

$$(5.8) \quad v(x) = \int e^{i|\xi|\varphi(x, y, \frac{\xi}{|\xi|})} a(x - y, \frac{\xi}{|\xi|}) v(y) \chi(y) dy d\xi.$$

Moreover, by the results of [HeSj2], we already know that, on the real,  $v$  cannot be exponentially large, that is, for any  $\varepsilon > 0$ , one has,

$$v = \mathcal{O}(e^{\varepsilon/h}) \text{ locally uniformly on } \mathbb{R}^n.$$

In addition,  $v$  is solution to

$$(5.9) \quad ((hD_x + \nabla\psi)^2 + V - \rho)v = 0.$$

We set,

$$Q(y, hD_y) := (hD_y + \nabla\psi)^2 + V(y) - \rho,$$

and, in order to estimate the integral, as in [Mal], we first plan to construct a symbol  $b = b(x, y, \tau, \xi, h) \sim \sum_{k \geq 0} b_k(x, y, \tau, h) |\xi|^{-k}$ , with large parameter  $|\xi|$ , in such a way that one has,

$$(5.10) \quad e^{-i|\xi|\varphi(x, y, \tau)} Q(y, hD_y) \left( e^{i|\xi|\varphi(x, y, \tau)} b \right) \sim a(x - y, \tau).$$

Here, the asymptotic must hold as  $|\xi| \rightarrow \infty$ , and the quantities  $\tau \in S^{n-1}$ ,  $\mu := \frac{1}{h|\xi|} \in (0, \frac{1}{C}]$  (with  $C > 0$  large enough) have to be considered as extra parameters. In particular, (5.10) can be rewritten as,

$$(5.11) \quad [(-D_y + \mu|\xi|\nabla\psi(y) - |\xi|\nabla_y\varphi)^2 + \mu^2|\xi|^2(V - \rho)] b \sim a(x - y, \tau) \mu^2 |\xi|^2.$$

Since  $(\nabla\psi)^2 = E_0 - V$  and  $\rho \rightarrow E_0$  as  $h \rightarrow 0_+$ , the (leading order) coefficient  $c_2$  of  $|\xi|^2$  satisfies,

$$\begin{aligned} c_2 &= (\mu\nabla\psi - \nabla_y\varphi)^2 + \mu^2(V - \rho) \\ &= (\nabla_y\varphi)^2 - 2\mu\nabla_y\varphi\nabla\psi + o(1) \\ &= (-\tau - i(x - y))^2 + 2\mu(\tau - i(x - y))\nabla\psi + o(1) \\ &= 1 + \mathcal{O}(|x - y| + \mu) + o(1). \end{aligned}$$

In particular, we can solve the transport equations for  $x, y$  close enough to  $z_0$ , and for  $\mu$  small enough, that is,  $|\xi| \geq C/h$  with  $C > 0$  sufficiently large. By the microlocal analytic theory of [Sj], we also know that the resulting formal symbol admits analytic estimates, and can therefore be re-summed into a symbol  $b(x, y, \tau, \xi, h)$  such that, for some constant  $\delta > 0$ , one has,

$$(5.12) \quad e^{-i|\xi|\varphi(x,y,\tau)t} Q(y, hD_y) \left( e^{i|\xi|\psi(x,y,\tau)} b \right) - a(x-y, \tau) = \mathcal{O}(e^{-\delta|\xi|}),$$

uniformly with respect to  $\tau \in S^{n-1}$ ,  $|\xi| \geq C/h$ ,  $h > 0$  small enough, and  $x, y \in \mathbb{C}^n$  close enough to  $z_0$ .

Then, splitting the integral in (5.8), we write,

$$(5.13) \quad v(x) = \int_{\{|\xi| \geq \frac{C}{h}\}} + \int_{\{|\xi| \leq \frac{\varepsilon_1}{h}\}} + \int_{\{\frac{\varepsilon_1}{h} \leq |\xi| \leq \frac{C}{h}\}}.$$

The first integral can be estimated by using (5.12), an integration by part, and the fact that  $v$  solves  $Qv = 0$ . One finds that it is  $\mathcal{O}(e^{-\delta_1/h})$  for some  $\delta_1 > 0$ .

The second integral can be estimated as in [Ma1], and it is  $\mathcal{O}(e^{\varepsilon_1|Imx|/h})$ .

For the third integral, we make the change of variable  $\xi = \eta/h$ , and we find,

$$(5.14) \quad h^{-n} \int_{\{\varepsilon_1 \leq |\eta| \leq C\}} e^{i(x-y)\eta/h - |\eta|(x-y)^2/2h} a(x-y, \frac{\eta}{|\eta|}) \chi(y) v(y) dy d\eta.$$

But, from the theory of [HeSj2], we know that if  $u$  is outgoing, then, near  $z_0$ , the microsupport of  $u$  satisfies,

$$MS(ue^{S_0/h}) \subset \Lambda_+.$$

Since  $\Lambda_+ = \{\nabla\psi(y); y \text{ close to } z_0\}$ , by standard rules on the microsupport we deduce,

$$MS(v) \subset \{\eta = 0\}.$$

As a consequence, the integral appearing in (5.14) is  $\mathcal{O}(e^{-\delta_2/h})$  for some  $\delta_2 > 0$ , and the result follows.  $\square$

Thanks to this proposition, we can enter the framework of the analytic pseudodifferential calculus of [Sj]. We set  $v := e^{-i\psi/h}u$ , and, in a complex neighborhood of  $z_0$ , we can write  $Pu = e^{-i\psi/h}Pe^{i\psi/h}v$  as,

$$(5.15) \quad Pu(x) = \frac{1}{(2\pi h)^n} \int_{\Gamma(x)} e^{i(x-y)\alpha_\xi/h - [(x-\alpha_x)^2 + (y-\alpha_x)^2]/2h} p_\psi(\alpha_x, \alpha_\xi) v(y) dy d\alpha,$$

where  $p_\psi$  is the symbol of  $P_\psi := e^{-i\psi/h}Pe^{i\psi/h}$ , and satisfies,

$$(5.16) \quad p_\psi(\alpha) = (\alpha_\xi + \nabla\psi(\alpha_x))^2 + V(\alpha_x) + \mathcal{O}(h),$$

and where  $\Gamma(x)$  is the (singular) complex contour of integration given by,

$$\Gamma(x) : \begin{cases} \alpha_\xi = 2i\varepsilon_1 \frac{\overline{x-y}}{|x-y|}; \\ |x-y| \leq r, \ y \in \mathbb{C}^n \text{ (} r \text{ small enough with respect to } \varepsilon_1 \text{)}; \\ |x-\alpha_x| \leq r, \ \alpha_x \in \mathbb{R}^n. \end{cases}$$

Let us observe that the identity (5.15) takes place in  $H_{-S_0+\varepsilon_1|\operatorname{Im} x|, z_0}$ , that is, modulo error terms that are exponentially smaller than  $e^{(-S_0+\varepsilon_1|\operatorname{Im} x|)/h}$  in a complex neighborhood of  $z_0$ .

Taking local coordinates  $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  near  $z_1$ , in such a way that  $dV(z_1) \cdot x = -cx_n$  with  $c > 0$  (and thus  $T_{z_1}\partial\mathcal{M} = \{x_n = 0\}$ ), we see that  $\nabla\psi(x)$  remains close to  $(0, \sqrt{x_n})$ . In particular, still working in these coordinates, the symbol  $-V(x) - (\xi' + \nabla_{x'}\psi(x))^2$  is elliptic along  $\Gamma(x)$  (at least if  $\varepsilon_1$  has been chosen sufficiently small), and with positive real part. Thus, in view of (5.16), so is  $(\xi_n + \partial_{x_n}\psi(x))^2 - p_\psi(x, \xi) + \rho$ . As a consequence, applying the symbolic calculus of [Sj], we conclude to the existence of a pseudodifferential operator  $A = A(x, hD_{x'})$ , with principal symbol,

$$a(x, \xi') = \sqrt{\rho - V(x) - (\xi' + \nabla_{x'}\psi(x))^2},$$

such that  $P_\psi - \rho$  can be factorised as,

$$P_\psi - \rho = (hD_{x_n} + \partial_{x_n}\psi(x) + A) \circ (hD_{x_n} + \partial_{x_n}\psi(x) - A),$$

when acting on  $H_{-S_0+\varepsilon_1|\operatorname{Im} x|, z_0}$ . Since  $(P_\psi - \rho)v = 0$ , and  $hD_{x_n} + \partial_{x_n}\psi(x) + A$  is elliptic along  $\Gamma(x)$ , we deduce,

$$(5.17) \quad (hD_{x_n} + \partial_{x_n}\psi(x) - A)v = 0 \quad \text{in } H_{-S_0+\varepsilon_1|\operatorname{Im} x|, z_0}.$$

Now, going back to (5.5), and choosing  $\Omega$  in such a way that its boundary contains  $z_0$  and is of the form  $\{x_n = \delta_0\}$  (with  $\delta_0 > 0$  constant) near  $z_0$ , the corresponding part of the integral can be written as,

$$I_0 := -h^2 \operatorname{Im} \int_{\{x_n=\delta_0\} \cap W_0} \left( \frac{\partial v}{\partial x_n} + \frac{i}{h} \frac{\partial \psi}{\partial x_n} v \right) \overline{v} dx'$$

where  $W_0$  is a small real neighborhood of  $z_0$ . Thus, using (5.17), we obtain,

$$I_0 = -h \operatorname{Re} \int_{\{x_n=\delta_0\} \cap W_0} (Av) \overline{v} dx' + \mathcal{O}(e^{-(2S_0+\varepsilon_0)/h}),$$

with  $\varepsilon_0 > 0$ . Finally, observing that the principal symbol of  $A$  is strictly positive in  $(z_0, 0)$ , and proceeding as in [Ma1], Section 2 (in particular,

considering the realisation on the real of  $A$ ), we can construct an elliptic pseudodifferential operator  $B$  of order 0, such that,

$$A = B^*B + \mathcal{O}(e^{-(S_0+\varepsilon)/h})$$

on  $L^2(\{x_n = 0\} \cap W_0)$  (with some  $\varepsilon > 0$ ). Finally, taking advantage of the ellipticity of  $B$ , we conclude, as in [Ma1], Lemma 2.3, that we have,

$$I_0 \leq -\frac{h}{C} \|v\|_{L^2(\{x_n=0\} \cap W_0)}^2 + \mathcal{O}(e^{-(2S_0+\varepsilon_0)/h}),$$

where  $C, \varepsilon_0$  are positive constants. Summing up all the contributions, and observing that, in the previous formula,  $v$  may be replaced by  $u$  (since  $\psi$  is real on the real and  $v = e^{-i\psi/h}u$ ), we have proved,

**Proposition 5.2.** *There exist  $C, \varepsilon_0 > 0$  such that,*

$$|\operatorname{Im} \rho| \geq \frac{h}{C} \|u\|_{L^2(\partial\Omega)}^2 - Ce^{-(2S_0+\varepsilon_0)/h},$$

*uniformly for  $h > 0$  small enough.*

From now on, we proceed by contradiction : We assume the existence of  $\varepsilon_1 > 0$  such that,

$$|\operatorname{Im} \rho| = \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}),$$

uniformly as  $h \rightarrow 0_+$  (possibly along a sequence of numbers only). By the previous proposition, this implies,

$$(5.18) \quad \|u\|_{L^2(\partial\Omega)} = \mathcal{O}(e^{-(S_0+\varepsilon_1)/h}).$$

## 6. PROPAGATION ACROSS $\partial\mathcal{M}$

The purpose of this section is to propagate the estimate (5.18) up to the boundary of  $\mathcal{M}$  and beyond. As in the previous section, we start by working locally near some point  $z_1 \in \mathcal{T}_1$ , and we observe that (5.18) actually implies that  $e^{S_0/h}u$  is exponentially small near any point  $z_1(t) := \Pi_x \exp tH_p(z_1, 0)$ , with  $|t| > 0$  small enough (this can be seen either by standard propagation, or more simply by the fact that the choice of  $\Omega$  can be changed without altering (5.18), as long as its boundary stays in  $\mathcal{M}$ ).

Near  $z_1$ ,  $u$  does not satisfy anymore sufficiently good a priori estimates that would permit us to use standard propagation (we only have  $u = \mathcal{O}(e^{(-\min(S_0, d_V(U, x)) + \varepsilon)/h})$  for all  $\varepsilon > 0$ , and  $d_V(U, x)$  takes values strictly

less than  $S_0$  near  $x_1$ ). However, we will profit from the fact that the difference between  $S_0$  and  $d_V(U, x)$  is controlled by  $\delta(x)^{3/2}$ , where  $\delta(x)$  is the usual distance between  $x$  and the caustic set  $\mathcal{C}$  where  $x \mapsto d(U, x)$  becomes singular (see [HeSj2]).

For this purpose, we will use explicit Carleman estimates, in a spirit similar to that of [DGM] (see also [KSU]).

By assumption 3, we know that  $\mathcal{T}_1$  is finite. We fix once for all  $z_1 \in \mathcal{T}_1$ , and we will first prove that  $e^{S_0/h}u$  is exponentially small near  $z_1$ .

As in the previous section, we take local coordinates  $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  centered at  $z_1$ , in such a way that  $dV(z_1) \cdot x = -cx_n$  with  $c > 0$ . We also consider a tubular neighborhood  $Z_\delta$  of  $z_1$  of the form,

$$Z_\delta := \{-\delta \leq x_n \leq \delta_0; |x'| \leq \delta_0\},$$

where  $\delta_0 > 0$  is fixed sufficiently small, and  $\delta \in (0, \delta_0)$  is a small parameter that we will possibly shrink later on. We divide the boundary of  $Z_\delta$  into,

$$\partial Z_\delta = \Sigma_\delta \cup \Sigma_0 \cup \Sigma',$$

with,

$$\Sigma_\delta := \{x_n = -\delta; |x'| \leq \delta_0\};$$

$$\Sigma_0 := \{x_n = \delta_0; |x'| \leq \delta_0\};$$

$$\Sigma' := \{-\delta \leq x_n \leq \delta_0; |x'| = \delta_0\}.$$

Note that, thanks to Assumption 3 and (5.3), we already know that, for  $\delta$  sufficiently small, there exists  $\varepsilon_0 > 0$  independent of  $\delta$  such that,

$$(6.1) \quad u = \mathcal{O}(e^{-(S_0 + \varepsilon_0)/h}) \quad \text{on } \Sigma',$$

and, using the equation  $Pu = \rho u$ , we obtain similar estimates on the derivatives of  $u$ , too. Moreover, by (5.18), we also have,

$$(6.2) \quad u = \mathcal{O}(e^{-(S_0 + \varepsilon'_0)/h}) \quad \text{on } \Sigma_0,$$

for some constant  $\varepsilon'_0 > 0$  (and the same for the derivatives of  $u$ ).

Finally, by the same techniques as in [Ma1], Section 2, we see that, near  $z_1$ , the distance  $d_V(x, \mathcal{M})$  satisfies,

$$d_V(x, \mathcal{M}) = \mathcal{O}(|x_n|^{3/2}),$$

and, by the triangle inequality, we also have,

$$d_V(U, x) \geq S_0 - d_V(x, \mathcal{M}).$$

As a consequence, there exists a constant  $c_0 > 0$  such that,

$$d_V(U, x) \geq S_0 - c_0|x_n|^{3/2}$$

and thus, for any  $\delta, \varepsilon > 0$  small enough, one has,

$$(6.3) \quad u = \mathcal{O}(e^{-(S_0 - c_0\delta^{3/2} - \varepsilon)/h}) \quad \text{on } \Sigma_\delta,$$

and similarly for the derivatives of  $u$ .

**Proposition 6.1.** *For  $\delta > 0$  sufficiently small, there exists  $\varepsilon_\delta > 0$  such that,*

$$\|u\|_{L^2(Z_\delta)} = \mathcal{O}(e^{-(S_0 + \varepsilon_\delta)/h}),$$

uniformly for  $h > 0$  small enough.

*Proof.* The proof relies on some explicit Carleman-type estimates, in a way rather similar to that of [DGM]. We set,

$$v := e^{\alpha(x_n + \delta)/h} u,$$

where  $\alpha > 0$  is fixed sufficiently small in order to have  $2\alpha\delta_0 < \min(\varepsilon_0, \varepsilon'_0)/2$  (here  $\varepsilon_0, \varepsilon'_0$  are those of (6.1)-(6.2)). The function  $v$  is solution to,

$$(-h^2\Delta + V - \rho - \alpha^2 + 2h\alpha\partial_{x_n})v = 0,$$

and, by (6.1)-(6.3), for any  $\varepsilon > 0$  and for some  $\varepsilon_1 > 0$ , we have,

$$(6.4) \quad \begin{aligned} \|v\|_{H^2(\Sigma_0 \cup \Sigma')} &= \mathcal{O}(e^{-(S_0 + \varepsilon_1)/h}); \\ \|v\|_{H^2(\Sigma_\delta)} &= \mathcal{O}(e^{-(S_0 - c_0\delta^{3/2} - \varepsilon)/h}). \end{aligned}$$

On the other hand, we have,

$$(6.5) \quad \begin{aligned} 0 &= \|(-h^2\Delta + V - \rho - \alpha^2 + 2h\alpha\partial_{x_n})v\|_{L^2(Z_\delta)}^2 \\ &= \|(-h^2\Delta + V - \rho - \alpha^2)v\|_{L^2(Z_\delta)}^2 + 4h^2\alpha^2\|\partial_{x_n}v\|_{L^2(Z_\delta)}^2 \\ &\quad + 4h\alpha \operatorname{Re} \int_{Z_\delta} (-h^2\Delta + V - \rho - \alpha^2)v \overline{\partial_{x_n}v} dx. \end{aligned}$$

We first prove,

**Lemma 6.2.** *There exists a constant  $C > 0$  such that, for all  $\delta > 0$  small enough,*

$$(6.6) \quad \operatorname{Re} \int_{Z_\delta} (-h^2\Delta + V - \rho - \alpha^2)v \overline{\partial_{x_n}v} dx \geq \frac{1}{C} \|v\|_{L^2(Z_\delta)}^2 - Ce^{-2(S_0 - c_0\delta^{3/2} - \varepsilon)/h}.$$

*Proof.* We set  $Q := 4h\alpha \operatorname{Re} \int_{Z_\delta} (-h^2\Delta + V - \rho - \alpha^2)v \overline{\partial_{x_n} v} dx$ . By integrations by part in  $x_n$ , we have,

$$\begin{aligned} Q &= 4h\alpha \operatorname{Re} \int_{\Sigma_0} (-h^2\Delta + V - \rho - \alpha^2)v \cdot \overline{v} dx' \\ &\quad - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} (-h^2\Delta + V - \rho - \alpha^2)v \cdot \overline{v} dx' \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} \partial_{x_n} (-h^2\Delta + V - \rho - \alpha^2)v \cdot \overline{v} dx \end{aligned}$$

and thus, by (6.4),

$$\begin{aligned} Q &= 4h\alpha \operatorname{Re} \left[ - \int_{\Sigma_\delta} (-h^2\partial_{x_n}^2 - h^2\Delta_{x'})v \cdot \overline{v} dx' - \int_{\Sigma_\delta} (V - \rho - \alpha^2)v \cdot \overline{v} dx' \right] \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} \partial_{x_n} (-h^2\partial_{x_n}^2 - h^2\Delta_{x'} + V - \rho - \alpha^2)v \cdot \overline{v} dx + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}) \\ &= 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2\partial_{x_n}^2 v \cdot \overline{v} dx' + 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2\Delta_{x'} v \cdot \overline{v} dx' - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} (V - \rho - \alpha^2)|v|^2 dx' \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} (-h^2\partial_{x_n}^2 - h^2\Delta_{x'} + V - \rho - \alpha^2)\partial_{x_n} v \cdot \overline{v} dx \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} (\partial_{x_n} V)v \cdot \overline{v} dx + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}). \end{aligned}$$

Then, using Green's formula in the  $x'$  variables, and again (6.4), we obtain,

$$\begin{aligned} Q &= 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2\partial_{x_n}^2 v \cdot \overline{v} dx' - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2|\nabla_{x'}|^2 dx' - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} (V - \rho - \alpha^2)|v|^2 dx' \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} -h^2\partial_{x_n}^3 v \cdot \overline{v} dx - 4h\alpha \operatorname{Re} \int_{Z_\delta} -h^2\Delta_{x'} \partial_{x_n} v \cdot \overline{v} dx \\ &\quad - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} (V - \rho - \alpha^2)\partial_{x_n} v \cdot \overline{v} dx \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} (\partial_{x_n} V)v \cdot \overline{v} dx + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}) \\ &= 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2\partial_{x_n}^2 v \cdot \overline{v} dx' - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2|\nabla_{x'}|^2 dx' \\ &\quad - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} (V - \rho - \alpha^2)|v|^2 dx' - 4h\alpha \operatorname{Re} \int_{Z_\delta} h^2\partial_{x_n}^2 v \partial_{x_n} \overline{v} dx \\ &\quad - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} h^2\partial_{x_n}^2 v \cdot \overline{v} dx' - 4h\alpha \operatorname{Re} \int_{Z_\delta} (\partial_{x_n} v) \overline{(-h^2\Delta_{x'} + (V - \rho - \alpha^2))v} dx \\ &\quad - 4h\alpha \operatorname{Re} \int_{Z_\delta} (\partial_{x_n} V)v \cdot \overline{v} dx + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}). \end{aligned}$$

By an integration by parts, we also see that,



$$\operatorname{Re} \int_{Z_\delta} \partial_{x_n}^2 v \partial_{x_n} \bar{v} dx = -\frac{1}{2} \int_{\Sigma_\delta} |\partial_{x_n} v|^2 dx' + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h})$$

and therefore,

$$\begin{aligned} Q = & -4h\alpha \int_{\Sigma_\delta} h^2 |\nabla_{x'}|^2 dx' - 4h\alpha \operatorname{Re} \int_{\Sigma_\delta} (V - \rho - \alpha^2) |v|^2 dx' \\ & + 2h\alpha \int_{\Sigma_\delta} h^2 |\partial_{x_n} v|^2 dx' - \bar{Q} - 4h\alpha \operatorname{Re} \int_{Z_\delta} (\partial_{x_n} V) v \cdot \bar{v} dx + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}). \end{aligned}$$

Using that  $\bar{Q} = Q$ , we finally obtain,

$$\begin{aligned} Q = & -2h^3 \alpha \|\nabla_{x'} v\|_{\Sigma_\delta}^2 + h^3 \alpha \|\partial_{x_n} v\|_{\Sigma_\delta}^2 - 2h\alpha \int_{Z_\delta} (\partial_{x_n} V) |v|^2 dx \\ & - 2h\alpha \operatorname{Re} \int_{\Sigma_\delta} (V - \rho - \alpha) |v|^2 dx' + \mathcal{O}(e^{-2(S_0+\varepsilon_1)/h}) \\ = & -2h\alpha \int_{Z_\delta} (\partial_{x_n} V) |v|^2 dx + \mathcal{O}(e^{-2(S_0-c_0\delta^{3/2}-\varepsilon)/h}), \end{aligned}$$

where  $\varepsilon > 0$  is arbitrarily small. Since  $\partial_{x_n} V(z_1) = -c < 0$ , we deduce the result of the lemma.  $\square$

Inserting (6.6) into (6.5), we obtain,

$$\|v\|_{L^2(Z_\delta)}^2 = \mathcal{O}(e^{-2(S_0-c_0\delta^{3/2}-\varepsilon)/h}),$$

and therefore, since  $2\alpha(x_n + \delta) \geq \alpha\delta$  on  $Z_{\delta/2} \subset Z_\delta$ ,

$$\|u\|_{L^2(Z_{\delta/2})}^2 = \mathcal{O}(e^{-(2S_0+\alpha\delta-2c_0\delta^{3/2}-2\varepsilon)/h}).$$

Observing that  $\alpha$  has been chosen independently of  $\delta$ , we obtain the result of Proposition 6.1 by taking  $\delta$  sufficiently small in order to have  $\alpha\delta > 2c_0\delta^{3/2}$ .  $\square$

Summing up, using Lemma 6.1 at each point of  $\mathcal{T}_1$  and (5.3), we obtain,

**Proposition 6.3.** *There exists of a neighborhood  $\mathcal{V}$  of  $\partial\mathcal{M}$ , and a constant  $\varepsilon_2 > 0$ , such that,*

$$\|u\|_{L^2(\mathcal{V})} = \mathcal{O}(e^{-(S_0+\varepsilon_2)/h}),$$

*uniformly for  $h > 0$  small enough.*

## 7. COMPLETION OF THE PROOF

From this point, the proof proceeds exactly as in [Ma1]. More precisely, if  $\tilde{P} = -h^2\Delta + V$  is the operator defined as in (3.2), with  $\tilde{V} = V$  near  $\ddot{O} \setminus \mathcal{V}'$  (where  $\mathcal{M} \subset \subset \mathcal{V}' \subset \subset \mathcal{V}$ ), we already know (see [HeSj2], Theorem 9.9) that the difference  $u - u_0$  satisfies,

$$\|u - u_0\|_{L^2(\ddot{O} \setminus \mathcal{V}')} = \mathcal{O}(e^{-(S_0 + \varepsilon_3)/h}),$$

for some constant  $\varepsilon_3 > 0$ . Then, applying Proposition 6.3, we deduce,

$$\|u_0\|_{L^2(\mathcal{V} \setminus \mathcal{V}')} = \mathcal{O}(e^{-(d_V(U, x) + \varepsilon_3)/h}),$$

with  $\varepsilon_4 > 0$ . At this point, we are in a situation absolutely similar to that of [Ma1]. In particular, the previous estimate can be propagated up to the well  $U$  along any minimal geodesic  $\gamma \in G$ , and as in [Ma1], Section 6, we obtain that for all  $x_1 \in \bigcup_{\gamma \in G} (\gamma \cap \partial U)$ , one has,

$$(x_1, 0) \notin MS(u_0),$$

where  $MS(u)$  stands for the microsupport of  $u$  as defined, e.g., in [Ma2] (it was called  $FS_a(u)$  in [Ma1]). Since, in addition,  $MS(u) \subset p^{-1}(0)$ , and,

$$(\partial U \times \mathbb{R}^n) \cap p^{-1}(0) \subset \{\xi = 0\},$$

we deduce,

$$MS(u) \cap \left( \bigcup_{\gamma \in G} (\gamma \cap \partial U) \times \mathbb{R}^n \right) = \emptyset,$$

and thus by standard properties of  $MS(u)$  (see, e.g., [Ma2]), we conclude to the existence of a neighborhood  $W$  of  $\bigcup_{\gamma \in G} (\gamma \cap \partial U)$  and of a positive constant  $\varepsilon_5 > 0$ , such that,

$$\|u\|_{L^2(W)} = \mathcal{O}(e^{-\varepsilon_5/h}),$$

uniformly for  $h > 0$  small enough. But this is in contradiction with Assumption [ND], and the proof of Theorem 4.1 is complete.

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